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THE COMPLETE PAPPUS HEXAGON.*

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1. Propositions 138 and 139, Book VII of Pappus Alexandrinus,† as translated by F. Hultsch reads thus:

"Iam his demonstratis ostendendum erit, si parallelae sint $\alpha\beta$ $\gamma\delta$, et in eas incidant quaedam rectae $\alpha\delta$ $\alpha\varsigma$ $\beta\gamma$ $\beta\varsigma$, quarum $\alpha\delta$ $\beta\gamma$ concurrant in μ, et a quovis rectae $\alpha\beta$ puncto inter α et β sumpto ducantur $\epsilon\gamma$ $\epsilon\delta$, quarum $\epsilon\gamma$ cum $\alpha\varsigma$ concurrant in η et $\epsilon\delta$ cum $\beta\varsigma$ in κ, rectam esse quae per $\eta\mu\kappa$ transit."

"At ne sint parallelae $\alpha\beta$ $\gamma\delta$, sed convergant in puncto ν; dico rursus rectam esse quae per $\eta\mu\kappa$ transit."

These theorems Pappus proved by proportion, the equal ratios being respectively between two lines and between the rectangles of two pairs of lines.

Salmon‡ has the same theorems stated thus:

"If ABC are three points of one line and $A'B'C'$ are three points of another line, then the intersections $BC'/B'C$, $CA'/C'A$, $AB'/A'B$ lie on a line."

Different other writers have Pappus's theorem in some wording. The most important mention of the simple case is by Rudolph Boeger,§ who gives it as a simple form, free from the idea of projective relations, of "*Das Sechseck in der Geometrie der Lage.*"

Some other papers directly or indirectly presenting perspective triangles are simply noted:

H. Schroeter: Math. Annalen 2:553—562.

J. Vâlyi: Archiv der Math. und Physik, 1882, Bd. 70, ss. 105—110; 1884. 2. R. II. T., ss. 230—234.

Rosanes: Ueber Dreiecke in persp. Lage. Math. Ann. 2:549.

Hess: Beitraege z. Theorie d. mehrfach persp. Dreiecke. Ibid. 28: 167.

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*Read before the American Mathematical Society, Chicago Section, April 30, 1907.

†Pappi Mathematicae Collectiones a Federico Commandino Urbinate, in Latin, in U. S Cong. Library. Also, by F. Hultsch in three volumes (Greek and Latin), Berolini, 1877.

‡Conic Sections:—6th Ed., §268, p. 246, Ex. 1.

§*Sechseck und Involution*, Hamburg Mitteilungen, Bd. III., 9, Feb. 1899, s. 387.

Third: Triangles triply in persp. Proc. Edinburg Math. Soc. XIX, p. 10.

L. Klug: Desmische Vierseiten-Systeme. Monatshefte (1903) XIV, s. 74.

M. Pasch: Ueber Vier-eck und seit. Math. Ann. 26:211—216.

Caporali: Memorie, pp. 236, 252.

Veronese: Sull' Hexagrammum mysticum. Lincei Mem. II, 1 (1877), p. 649.

2. The complete figure is constructed thus:

Take six numbers 1—6 and regard them as the names of points or of lines, such that 1, 3, 5 and 2, 4, 6 are three and three respectively on a line or on a point.

We consider the cross-joins as follows:

$$\begin{array}{cccccc} 1&2&3&4&5&6\\ 3&2&5&4&1&6\\ 5&2&1&4&3&6 \end{array} \qquad \begin{matrix}(1)&(2)&(3)\\ (4)&(5)&(6)\\ (7)&(8)&(9)\end{matrix} \qquad \left.\begin{matrix}{[1]}\\ {[3]}\\ {[5]}\end{matrix}\right\}\text{on } \Sigma_1 \text{ or } D_1\text{, respectively.}\quad (1),$$

$$\begin{array}{cccccc} 3&2&1&4&5&6\\ 5&2&3&4&1&6\\ 1&2&5&4&3&6 \end{array} \qquad \begin{matrix}(10)&(11)&(12)\\ (13)&(14)&(15)\\ (16)&(17)&(18)\end{matrix} \qquad \left.\begin{matrix}{[2]}\\ {[4]}\\ {[6]}\end{matrix}\right\}\text{on } \Sigma_2 \text{ or } D_2\text{, respectively.}\quad (2).$$

When the six numbers name points, they are given as the points π_i. In that case, [1], etc., are lines to be known as P_i, where $i=1, 2, \ldots, 6$. In the dual case, the given six are lines P_i and [1] are points. The scheme means, when points are given, that the line of $\overline{1,2}$ intersects the line of $\overline{4,5}$ in a point (1), that $\overline{2,3}/\overline{5,6}$ is the point (2), and that $\overline{3,4}/\overline{6,1}$ is (3); further, that the points (1), (2), (3) lie on a line [1], and that three lines similarly gotten lie on the point Σ_1. Dualistically, we start with lines $P_{1,3,5}$ on Σ_1 and $P_{2,4,6}$ on Σ_2 and close with points π_i, three on each line D_1 and D_2.

Following the names given to points and lines in the Pascal Hexagon, the points Σ are called Steiner points; the lines P_1, Pappus lines; and the lines D we call Hessian diagonals.

3. From the complete Pappus Hexagon we have the following theorems:

I.

Three lines P_i on each of two points Σ_1, Σ_2, joined by the line S, intersect crosswise in nine points α_{1-3}, β_{1-3}, γ_{1-3}, which join by 18 lines, which are the sides of two sets of three point-triads each.	Three points π_i on each of two lines D_1, D_2, meeting in the point δ, are cross-joined by nine lines which meet in eighteen points, 1—18, which are the vertices of two sets of three line-triads each.

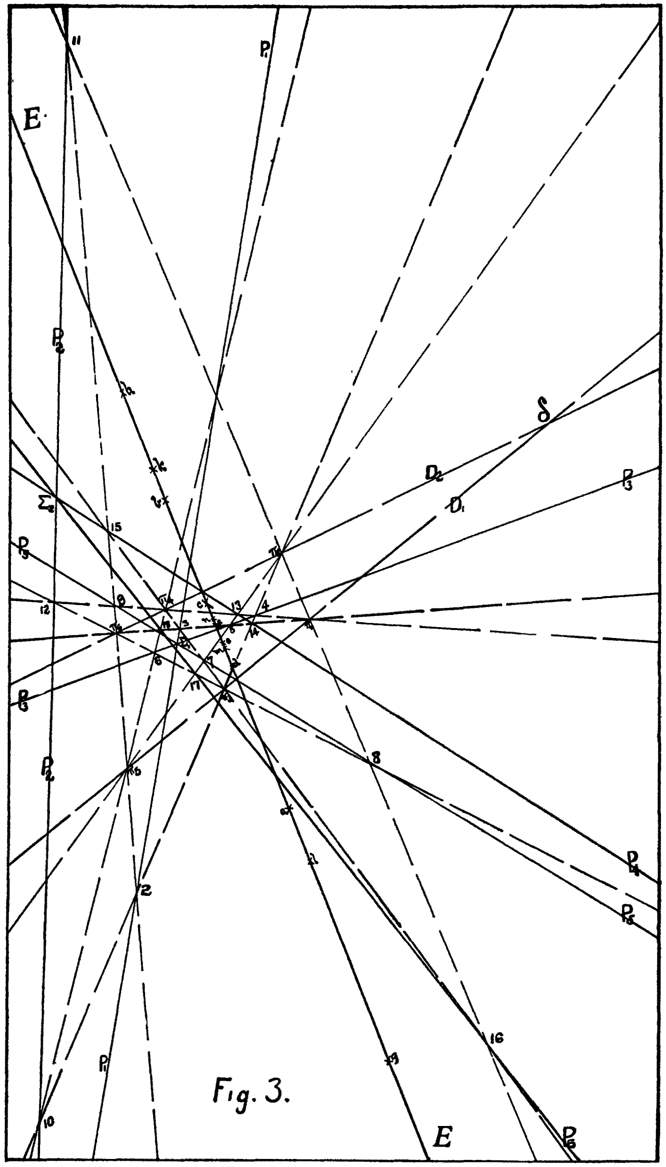

Fig. 1.

The triangles of each set are independently by twos in triple perspective, having as centers of perspective the points Σ_1 and Σ_2 each three lines and three points π_i on the line D_2 or D_1, respectively; and

The triangles of each set are independently by twos in triple perspective, having as axes of perspective the lines D_1 and D_2 each three times and three lines P_i on the point Σ_2 or Σ_1, respectively; and having

having as axes of perspective the line D_1 or D_2 three times each for the sets, respectively, and twelve other lines all of which pass through a point ϵ, which is the pole of the line S as to any of the six triangles. The 18 lines above lie by three on six points π_i, which are three and three on the lines D above.

as axes of perspective the point Σ_1 or Σ_2 three times each for the sets, respectively, and twelve other points all of which lie on a line E, which is the polar of the point δ as to any of the six triangles. The 18 points above lie by three on six lines P_i, which are three and three on the points Σ above.

The frame-work of this theorem as to the triangles and their being in triple perspective is not new but seems a necessary preface to the new parts as to the covariant point ϵ, and line E, and the complete duality that makes the figures really one whole since starting with the three points π_i on each line D_1 and D_2 of the left-hand theorem and following out the right-hand theorem we come back to the original points Σ_1 and Σ_2.

II.

The lines D_1, D_2, and S are the false sides of the complete quadrilateral of the Hessian pairs of the line-triads P_i on Σ_1 and Σ_2.

The points Σ_1, Σ_2, and δ are the false vertices of the complete quadrangle of the Hessian pairs of the point-triads π_i on D_1 and D_2.

From these theorems as also from the demonstration of them, follows the general theorem:

III.

Three lines on each of two points give rise to three points on each of two lines, and the latter by reciprocating the process give rise to three lines on each of the original two points. The derived three lines have the same Hessian pair, or are inclined to each other at the same angles as the original three.

4. We now give a proof of the theorems on the left. Take the two Steiner points, Σ_1, Σ_2, to have co-ordinates σ_1, σ_2, σ_3 and s_1, s_2, s_3, respectively; the triangle $\alpha_1\alpha_2\alpha_3$, formed by the intersections of the Pappus lines as will be shown, as reference triangle; and the line S as auxiliary line.

The line S determined by Σ_1 and Σ_2 is given by

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = 0, \text{ and will be written}$$

$$S_1x_1+S_2x_2+S_3x_3=0.$$

Since this line is taken as auxiliary line, we have the following relations:

$$\left.\begin{array}{c} S_1=S_2=S_3=1, \quad \sigma_1+\sigma_2+\sigma_3=0, \quad s_1+s_2+s_3=0, \\ s_i^2\sigma_j\,\sigma_k-s_j\,s_k\,\sigma_i^2=\sigma_k\,s_k-\sigma_j\,s_j, \end{array}\right\} \tag{3}$$

where i, j, k are each 1, 2, 3, successively.

The equations of the Pappus lines are the corresponding minors as represented thus:

$$\begin{array}{ccc} P_1 & P_3 & P_5 \\ \hline \sigma_1 & \sigma_2 & \sigma_3 \\ x_1 & x_2 & x_3 \end{array} \qquad \begin{array}{ccc} P_2 & P_4 & P_6 \\ \hline s_1 & s_2 & s_3 \\ x_1 & x_2 & x_3 \end{array} \tag{4}$$

The nine intersections of the six lines P other than Σ_1 and Σ_2 are named thus:

$$\begin{array}{lccl} \text{Line } & P_1 & P_5 & P_3 \\ \cline{2-4} \text{meets line } & P_2 & P_4 & P_6, \text{ respectively, in point} \\ & P_4 & P_6 & P_2, \\ & P_6 & P_2 & P_4, \end{array} \quad \left.\begin{array}{ccc} \alpha_2 & \alpha_3 & \alpha_1 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_3 & \gamma_1 & \gamma_2 \end{array}\right\} \tag{5}$$

From these co-ordinates by carrying out scheme (1) we find the co-ordinates of the points π_1, π_3, π_5, and of the line D_1 on which they lie. Likewise, following (2) and remembering the relations (3), we find the co-ordinates of the points π_2, π_4, π_6 and of their line D_2. From the co-ordinates of the lines D_1 and D_2 we write the co-ordinates of δ their intersection.

The 9 lines which by threes pass through the points π_i on D_1 are the sides of three point-triads

$$\left\{\begin{array}{cccc} (1) & \alpha_1 & \alpha_2 & \alpha_3, \\ (2) & \beta_1 & \beta_2 & \beta_3, \\ (3) & \gamma_1 & \gamma_2 & \gamma_3; \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{cccc} (1') & \alpha_1 & \beta_1 & \gamma_1, \\ (2') & \alpha_2 & \beta_2 & \gamma_2, \\ (3') & \alpha_3 & \beta_3 & \gamma_3, \end{array}\right\}$$

are the second set of three triangles whose sides pass by threes through the points π_{2i} on D_2.

The triangles of each set are found to be in triple perspective as indicated thus:

Triangles.	*Centers.*	*Axes of Perspective.*
$\left.\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{array}\right\}$	π_2, Σ_2, Σ_1.	D_1, $(\sigma_2, \sigma_3, \sigma_1)$, (s_3, s_1, s_2).
$\left.\begin{array}{ccc} \alpha_1 & \alpha_2 & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array}\right\}$	π_4, Σ_1, Σ_2.	D_1, (s_2, s_3, s_1), $(\sigma_3, \sigma_1, \sigma_2)$.
$\left.\begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{array}\right\}$	π_6, Σ_2, Σ_1.	D_1, $(\sigma_1, \sigma_2, \sigma_3)$, (s_1, s_2, s_3).

$$\left.\begin{matrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\end{matrix}\right\} \qquad \pi_1,\ \Sigma_2,\ \Sigma_1. \qquad D_2,\ (\sigma_2,\ \sigma_1,\ \sigma_3),\ (s_2,\ s_1,\ s_3).$$

$$\left.\begin{matrix}\alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{matrix}\right\} \qquad \pi_3,\ \Sigma_2,\ \Sigma_1. \qquad D_2,\ (s_3,\ s_2,\ s_1),\ (\sigma_3,\ \sigma_2,\ \sigma_1).$$

$$\left.\begin{matrix}\alpha_3 & \beta_3 & \gamma_3\\ \alpha_1 & \beta_1 & \gamma_1\end{matrix}\right\} \qquad \pi_5,\ \Sigma_1,\ \Sigma_2. \qquad D_2,\ (\sigma_1,\ \sigma_3,\ \sigma_2),\ (s_1,\ s_3,\ s_2).$$

The twelve axes other than the D's, evidently pass through a point with co-ordinates (1, 1, 1), which is thus the auxiliary point ϵ, the pole S as to the reference triangle. Since,

S has the same pole, ϵ, as to all point-cubics consisting of three joins of the six lines P_i, two and two,

ϵ is the pole of S as to every and any of the six triangles on all six lines P_i.

δ has the same polar, E, as to all line-cubics consisting of three joins of the six points π_i, two and two,*

E is the polar of δ as to every and any of the six triangles on all six points π_i.

THE LINES D_1 AND D_2 AS DIAGONALS OR FALSE SIDES OF THE COMPLETE QUADRILATERAL OF THE HESSIAN PAIRS.

5. The following linear relation exists between the three lines P_i on Σ_1:

$$\sigma_1(\sigma_2 x_3-\sigma_3 x_2)+\sigma_2(\sigma_3 x_1-\sigma_1 x_3)+\sigma_3(\sigma_1 x_2-\sigma_2 x_1)=0.$$

The Hessian covariant of the binary cubic is given by the sum of the squares of these three terms separately, and the imaginary Hessian lines are these terms with the respective coefficients 1, ω, ω^2, and 1, ω^2, ω, in the second case. These two lines and the analogous two on Σ_2 reduce to

$$\sigma_2\sigma_3 x_1+\omega\sigma_3\sigma_1 x_2+\omega^2\sigma_1\sigma_2 x_3=0, \qquad \{1\}$$

$$\sigma_2\sigma_3 x_1+\omega^2\sigma_3\sigma_1 x_2+\omega\sigma_1\sigma_2 x_3=0, \qquad \{2\}$$

The same as {1} with s for σ, {3}

The same as {2} with s for σ, {4}.

These intersect as follows:

{1} and {3} in imaginary point I: $(\sigma_1 s_1,\ \omega^2\sigma_2 s_2,\ \omega\sigma_3 s_3)$.

{2} and {4} in imaginary point J: (same with ω and ω^2 interchanged).

{1} and {4} in imaginary point H: $[(\sigma_1 s_1(\sigma_2 s_3-\omega\sigma_3 s_2),\ \sigma_2 s_2(\sigma_3 s_1-\omega\sigma_1 s_3),\ \sigma_3 s_3(\sigma_1 s_2-\omega\sigma_2 s_1)]$.

{2} and {3} in imaginary point K: [same with ω^2 for ω].

*Salmon: *Higher Plane Curves*, §166, pp. 143, 144.

Whence we find that the line of $\overline{IJ}$ is D_1, and that of $\overline{HK}$ is D_2; therefore, D_1 and D_2 are the diagonals of the imaginary quadrilateral of the Hessian pairs of the line-triads on Σ_1 and Σ_2.

6. As may be seen from Fig. 1, for the reverse process we have the triply perspective triangles, thus:

Triangles.	*Centers.*	*Axes of Perspective.*
1 6 8 / 2 4 9	$\Sigma_1, a, b.$	$P_2, D_2, D_1.$
1 6 8 / 7 3 5	$c, \Sigma_1, d.$	$D_2, P_6, D_1.$
2 4 9 / 5 7 3	$e, f, \Sigma_1.$	$D_2, D_1, P_4.$
10 14 18 / 16 11 15	$g, \Sigma_2, h.$	$D_2, P_1, D_1.$
11 15 16 / 17 19 13	$k, \Sigma_2, l.$	$D_2, P_5, D_1.$
10 14 18 / 12 13 17	$\Sigma_2, m, n.$	$P_3, D_2, D_1.$

7. The special forms or arrangements for the three lines on each of two points, to which attention is called, are:

(1) Two sets of equispaced triads, *i. e.*, lines at 120°.

(a) The points being the two equiangular points of the triangles of one set of three.

(b) The points on the circumcircle of equiangular triads.

(2) The points taken at infinity,

(a) Arbitrarily, giving two sets of three parallel lines each, at an angle D_1 with each other.

(b) At I and J, the circular imaginary points, giving two sets of perpendicular lines.

The several forms are handled best by using different co-ordinate systems most convenient for the particular case.

A special form of three points on each of two lines is had when three of the nine lines cross-joining the points by pairs meet in a point either within or without the acute angle at δ. Evidently one triangle of one or of the other set of three becomes a point, which point is then also Σ_1 or Σ_2 and the first six or the second six of the twelve points on E, which thus passes through the same point. All this follows readily in the analysis by observing the relation existing between the co-ordinates when three of the vertices of one triangle are equated.

8. Without further proof because they follow from data already given and may be tested in Fig. 1, we present two more theorems and two others relative to the special forms above.

IV.

A triangle of one set of three is in two-fold perspective with any one of the opposite set of three triangles, but for all such perspectivities—

there are only nine axes each taken twice and situated on the covariant point ϵ. The centers in each case are Σ_2 and Σ_1.

there are only nine centers each taken twice and situated on the covariant line E. The axes in each case are D_2 and D_1.

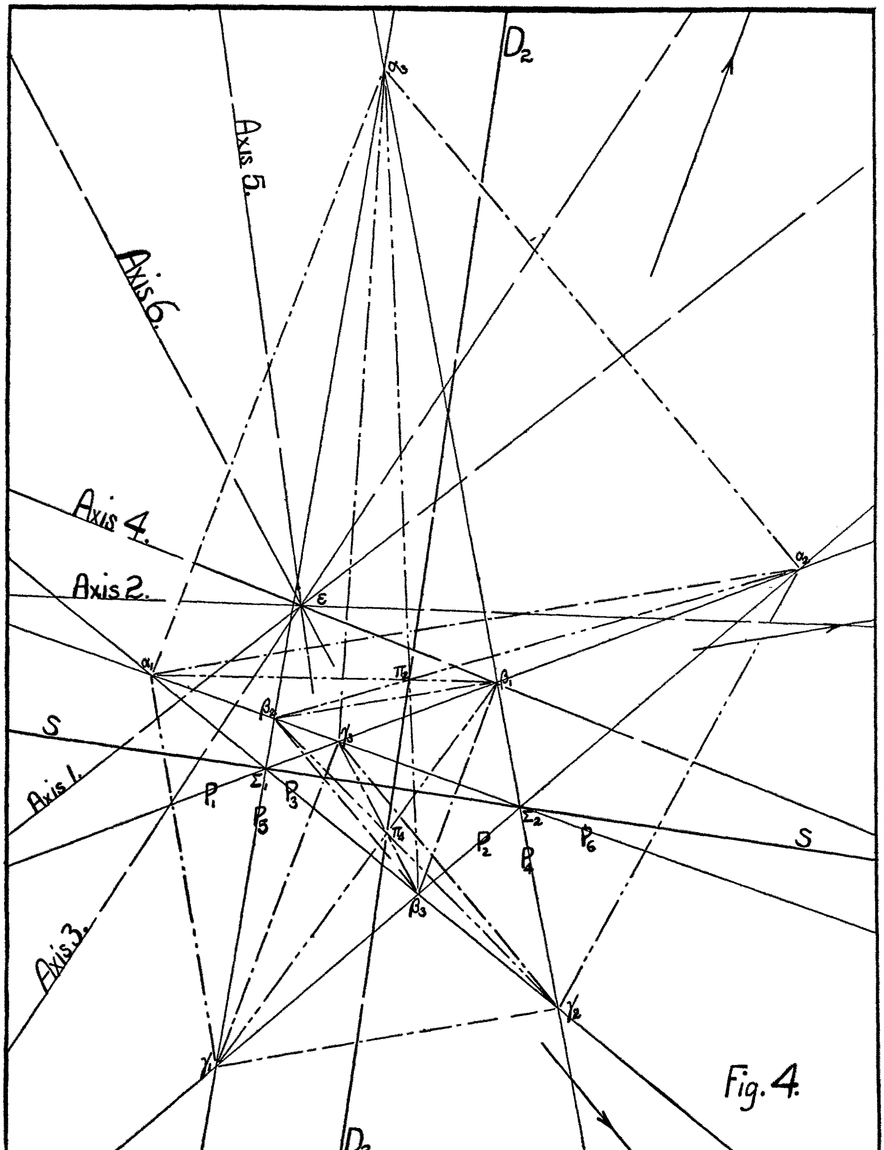

Fig. 2.

V.

The point- and line-triads are between themselves in *single* perspective. The center of perspective in each case is Σ_1 if the two triads are both of the first or both of the second set of three as herein classified, and Σ_2 is center if they are of oppositely named sets.

VI.

For a triad of equispaced lines on each of two points, one set of the three point-triads consists of equiangular triangles with sides respectively parallel (Fig. 2). Thus one of the lines D is at infinity and the other is perpendicular bisector of the line S between Σ_1 and Σ_2. Furthur, the circumcircles of the three equilateral triads pass through Σ_1 and Σ_2, and those of the other set of three triangles intersect in ϵ, the pole of S as to any of the six triangles.

VII.

If to the conditions of the previous theorem we add that one of the set of scalene triangles is also equiangular, then the vertices of the other two triangles of its set are inverse points as to its circumcircle, and all the circumcenters of the set of three equilateral triangles are on the finite Hessian diagonal D.

The proof of this last theorem is especially neat by use of circular co-ordinates.

Using Conjugate Co-ordinates.

9. As a convenient projection of the hexagon, we take the points π_2, π_4, π_6 on a line, considered the axis of reals, and the points π_1, π_3, π_5 on the line at infinity, so that the lines from the three points on the axis to these three are equispaced lines, parallel, respectively, to

$$x=ty \qquad x=\omega ty \qquad x=\omega^2 ty$$

going, respectively, to π_1 $\qquad$ π_3 $\qquad$ π_5

$$\begin{array}{llll}
\text{Lines on } \pi_2\equiv a{:}x=ty-a(t-1). & x=\omega ty-a(\omega t-1). & x=\omega^2 ty-a(\omega^2 t-1). & \\
\text{Lines on } \pi_4\equiv b{:}x=ty-b(t-1). & x=\omega ty-b(\omega t-1). & x=\omega^2 ty-b(\omega^2 t-1). & (6)\\
\text{Lines on } \pi_6\equiv c{:}x=ty-c(t-1). & x=\omega ty-c(\omega t-1). & x=\omega^2 ty-c(\omega^2 t-1). &
\end{array}$$

The points (p), for $p=1, 2, \ldots, 9$, are in general according to the scheme above,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x=\frac{a-b\,\omega-(a-b)\,\omega^{i+j}t}{1-\omega};$$

and the points (q), where $q=10, 11, \ldots, 18$, are,

$$\pi_{i,a} \text{ with } \pi_{j,b} \text{ is } x=\frac{a-b\,\omega^2-(a-b)\,\omega^{i+j-1}t}{1-\omega^2},$$

or are got from points (p) by interchanging b and c in the equation where $p=q-9$. The same interchange holds throughout the paragraph.

In the above a and b each permute for a, b, c, but a is never b; and ij are each 1, 3, 5, but $i\neq j$ in any one equation. As indicated above a, b, c are the circular co-ordinates of π_2, π_4, π_6 along D_2, the axis of reals.

These 18 points in 6 sets of 3 each, as indicated in schemes (1) and (2), lie on six lines P_i. From the equations of points (p) we get the co-ordinates of lines P.

The lines P_i, for $i=1, 3, 5$, we have the co-ordinates

$$P_i : a+b\omega+c\omega^2, \quad -(a+b\omega^2+c\omega)\omega^{-i}t,$$
$$\omega^2(ab+bc\omega+ca\omega^2)-(ab+bc\omega^2+ca\omega)\omega^{-i+1}t.$$

The co-ordinates of lines P_j, for $j=2, 4, 6$, are

$$P_j : a+b\omega^2+c\omega, \quad -(a+b\omega+c\omega^2)\omega^{-j+1}t,$$
$$\omega(ab+bc\omega^2+ca\omega)-(ab+bc\omega+ca\omega^2)\omega^{-j}t.$$

These lines are evidently equispaced on Σ.

The three lines P_i lie on the point Σ_1, which is

$$x=-\omega^2\frac{ab+bc\omega+ca\omega^2}{a+b\omega+c\omega^2};$$

and the lines P_j lie on the point Σ_2,

$$x=-\omega\frac{ab+bc\omega^2+ca\omega}{a+b\omega^2+c\omega}.$$

These two points are evidently conjugate and therefore symmetrical as to the axis of reals. Further, since the axis of reals, D_2, bisects the line between Σ_1 and Σ_2, the two points Σ are harmonic as to the two lines D.

Since the points Σ are independent of t, these points remain the same for any three equispaced lines on a, b, and c, respectively, mutually parallel; or, keeping one triad of points fixed the triad on the other line may move all along that line provided only that the angles between the lines to them remain constant.

Reversing the Process.

10. The lines P_{ij} intersect in nine points, called before α_{1-3}, β_{1-3}, γ_{1-3}, as follows:

$$P_1P_2 : x=\frac{a^2-bc+(a-b)(a-c)\omega^2t}{2a-b-c}.$$

$$P_1P_4 : x = \frac{c^2 - ab + (c-a)(c-b)\omega t}{2c-a-b}.$$

$$P_{16} : x = \frac{b^2 - ca + (b-c)(b-a)t}{2b-c-a}.$$

The remaining may be written easily by comparing these with the following:

	P_1	P_2	P_3
P_2	a^2, ω^2	b^2, ω	$c^2, 1$
P_4	c^2, ω	$a^2, 1$	b^2, ω^2
P_6	$b^2, 1$	c^2, ω^2	a^2, ω

The lines joining these intersections in pairs as indicated by schemes (1) and (2) meet by threes in points along D_2 and D_1.

The lines (q) to the *new* points along D_1 have slopes, respectively, t^2, ωt^2, $\omega^2 t^2$; so they turn twice the angle from the axis as the original lines and are like them equispaced.

The lines (p) intersect by threes on three new points, π'_1, π'_3, π'_5, along D_2, corresponding, respectively, with a, b, c if we consider the external segment of D_2. They are

$$x = \frac{\begin{vmatrix} a-b & b^2-ca \\ c-a & c^2-ab \end{vmatrix}}{\begin{vmatrix} a-b & 2b-c-a \\ c-a & 2c-a-b \end{vmatrix}}; \text{ etc.,}$$

taking a, b, c in cyclic order.

11. Thus far the origin on the axis has been arbitrary. Now, considering it the centroid of the three given points, we have $a+b+c=0$, whence also

$$a^2-bc=b^2-ca=c^2-ab=\lambda, \text{ say; } bc+ca+ab=-\lambda;\ 2a-b-c=3a, \text{ etc.}$$

The three new points then become, respectively,

$$x = \frac{a\lambda}{-2bc+ca+ab}, \quad x = \frac{b\lambda}{bc-2ca+ab}, \quad x = \frac{c\lambda}{bc+ca-2ab}.$$

The counter-triad of the three points a, b, c is

$$\frac{-2bc+ca+ab}{3a} \text{ from } (xa/bc) = -1; \text{ call it } a'.$$

$$\frac{bc-2ca+ab}{3b} \text{ from } (xb/ca) = -1; \text{ call it } b'.$$

$$\frac{bc+ca-2ab}{3c} \text{ from } (xc/ab) = -1, \text{ call it } c'.$$

The cubic along the line a, b, c is $x^3-\lambda x-abc=0$, Differentiating this as to x, we have $3x^2-\lambda=0$, the roots of which are the polar pair of infinity, the intersection of the two lines D_1 and D_2. Calling the roots f and f', we have

$$f+f'=0, \text{ and } ff'=-f^2=-f'^2=-\tfrac{1}{3}\lambda. \quad \therefore f^2=f'^2=\tfrac{1}{3}\lambda.$$

By observation we see then that

$$\pi'_1\ a'=\pi'_3\ b'=\pi'_5\ c'=\tfrac{1}{3}\lambda;$$

so the new triad and the counter-triad of the original triad are in an involution whose double points are the polar pair of the intersection of the lines D_1 and D_2 as to the original triad.

From the results in § 9—11, we see that *in this form* by revolving the three lines on Σ, keeping the triad equispaced, we cut out, at each instant, along D_2 a triad of points having the same Hessian pair and being in the same involution whose double-points are the polar pair of the intersection of the Hessian diagonals as to the original triad.

Thus, we generate a pencil of point-triads along each of two lines, and dualistically of line-triads on each of two points.

THE TRISECTION PROBLEM.

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The solution of this problem by means of the quadratrix, conchoid, and the cardioid are well known, and statements of the fact that the problem has been solved by means of the hyperbolic curve are equally well known, but the writer has never seen a solution by the last named method.

Ball, in his History of Mathematics, says that Viviani solved the problem by means of the *equilateral* hyperbola, and that Vieta determined that its solution depends upon the solution of a cubic equation.

I am not aware that the solution by means of the ceroid [so called from its resemblance to a pair of horns] has ever before been given.

I. Solution by Means of the Hyperbolic Curve.

If a series of circles be drawn through two points A and B, and if BP be one third of the arc BPA and H and H' points in the perpendicular bisector of AB, the locus of the point P, as the circle varies in size, is an hyperbola, since $PB=2PH$ constantly.

As the curve is central, its general equation is